

A note on the Hamiltonian formalism for higher-derivative theories

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Abstract

An alternative version of Hamiltonian formalism for higher-derivative theories is presented. It is related to the standard Ostrogradski approach by a canonical transformation. The advantage of the approach presented is that the Lagrangian is nonsingular and the Legendre transformation is performed in a straightforward way.

*supported by the grant 691 of University of Lodz and European Social Fund and Budget of State under the Integrated Regional Operational Programme

[†]supported by the grants 690 and 795 of University of Lodz

[‡]supported by the grant 1 P03B 02 128 of the Polish Ministry of Science.

In some theories the Lagrangians containing higher time derivatives appear naturally. This concerns effective low energy theories, modified theories of gravity or noncommutative field theories. A standard framework for dealing with such theories on hamiltonian level is provided by Ostrogradski formalism [1], [2], [3]. The main disadvantage of the latter is that the Hamiltonian, being linear function of some momenta, is unbounded from below. In general, this cannot be cured by trying to devise an alternative canonical formalism. In fact, any Hamiltonian is an integral of motion while it is by far not obvious that a generic system described by a higher-derivative Lagrangian possesses globally defined integrals of motion except the one related to time translation invariance.

Ostrogradski approach has also some technical disadvantages. There is no straightforward transition from the Lagrangian to the Hamiltonian formalism. On the contrary, one has to introduce first the Lagrange multipliers enforcing the proper relations between some coordinates and the time derivatives of other coordinates; then the Dirac formalism of constrained dynamics is used to construct the Hamiltonian [2], [3], [4].

Ostrogradski approach is based on the idea that the consecutive time derivatives of the initial coordinate form new coordinates, $q_i \sim q^{(i-1)}$. However, it has been suggested [5], [6], [7] that one can use every second derivative as a new variable, $q_i \sim q^{(2i-2)}$. In the present note we study this idea in some detail. Following Ref. [5] we modify the initial Lagrangian by adding a term which on-shell becomes a total time derivative. It appears that part of new coordinates can be identified with even time derivatives of the initial one. The resulting Lagrangian is nonsingular and the Legendre transformation can be easily performed. The Hamiltonian coincides with the Ostrogradski one while the canonical variables are related to the standard ones by a canonical transformation. The generating function for this transformation is given explicitly.

We start with the Lagrangian depending on time derivatives up to some even order

$$L = L(q, \dot{q}, \ddot{q}, \dots, q^{(2n)}) \quad (1)$$

Define new variables

$$\begin{aligned} q_i &\equiv q^{(2i-2)}, & i = 1, \dots, n+1 \\ \dot{q}_i &\equiv q^{(2i-1)}, & i = 1, \dots, n \end{aligned} \quad (2)$$

so that

$$L = L(q_1, \dot{q}_1, q_2, \dot{q}_2, \dots, q_n, \dot{q}_n, q_{n+1}) \quad (3)$$

Let further F be any (at least twice differentiable) function of the following variables

$$F = F(q_1, \dot{q}_1, \dots, q_n, \dot{q}_n, q_{n+1}, q_{n+2}, \dots, q_{2n}) \quad (4)$$

obeying

$$(i) \quad \frac{\partial L}{\partial q_{n+1}} + \frac{\partial F}{\partial \dot{q}_n} = 0 \quad (5)$$

and

$$(ii) \quad \det \left[\frac{\partial^2 F}{\partial q_i \partial \dot{q}_j} \right]_{\substack{n+2 \leq i \leq 2n \\ 1 \leq j \leq n-1}} \neq 0, \quad n \geq 2 \quad (6)$$

(for $n = 1$ only (i) remains).

Finally, we define a new Lagrangian

$$\mathcal{L} \equiv L + \sum_{k=1}^n \left(\frac{\partial F}{\partial q_k} \dot{q}_k + \frac{\partial F}{\partial \dot{q}_k} q_{k+1} \right) + \sum_{j=n+1}^{2n} \frac{\partial F}{\partial q_j} \dot{q}_j \quad (7)$$

Let us have a look on Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = 0, \quad i = 1, \dots, 2n \quad (8)$$

Using (3), (4), (7) and (8) one finds

$$\sum_{k=1}^n \frac{\partial^2 F}{\partial q_i \partial \dot{q}_k} (q_{k+1} - \ddot{q}_k) = 0 \quad i = n+1, \dots, 2n \quad (9)$$

However, eqs. (3), (4) and (5) imply that $\frac{\partial^2 F}{\partial q_i \partial \dot{q}_n} \neq 0$ only for $i = n+1$. Therefore, (6) and (9) give

$$q_{k+1} = \ddot{q}_k, \quad k = 1, \dots, n \quad (10)$$

Let us now consider (8) for $1 \leq i \leq n$. We find

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) + \frac{\partial F}{\partial \dot{q}_{i-1}} - \frac{d^2}{dt^2} \left(\frac{\partial F}{\partial \dot{q}_i} \right) = 0, \quad i = 1, \dots, n \quad (11)$$

where, by definition $\frac{\partial F}{\partial \dot{q}_0} = 0$. By combining these equations and using (5) and (10) we arrive finally at the initial Lagrange equation

$$\sum_{k=0}^{2n} (-1)^k \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial q^{(k)}} \right) = 0 \quad (12)$$

Let us now consider the Hamiltonian formalism. Contrary to the Ostrogradski approach the Legendre transformation can be immediately performed; neither additional Lagrange multipliers nor constraints analysis are necessary. In fact, let us define the canonical momenta in a standard way

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad (13)$$

so that

$$p_i = \frac{\partial F}{\partial q_i}, \quad i = n+1, \dots, 2n \quad (14)$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} + \sum_{k=1}^n \left(\frac{\partial^2 F}{\partial \dot{q}_i \partial q_k} \dot{q}_k + \frac{\partial^2 F}{\partial \dot{q}_i \partial \dot{q}_k} q_{k+1} \right) + \sum_{j=n+1}^{2n} \frac{\partial^2 F}{\partial \dot{q}_i \partial q_j} \dot{q}_j + \frac{\partial F}{\partial q_i}, \quad i = 1, \dots, n \quad (15)$$

By virtue of eqs.(5) and (6) eqs.(14) can be solved for $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$

$$\dot{q}_i = f_i(q_1, \dots, q_{2n}, p_{n+1}, \dots, p_{2n}), \quad i = 1, \dots, n \quad (16)$$

Using again eqs.(5) and (6) we can now solve (15) for \dot{q}_i , $n+1 \leq i \leq 2n$. Finally, the Hamiltonian is calculated according to the standard prescription.

In order to compare the present formalism with the Ostrogradski approach let us note that they must be related by a canonical transformation. To see this we define new (Ostrogradski) variables \tilde{q}_k, \tilde{p}_k , $1 \leq k \leq 2n$:

$$\tilde{q}_{2i-1} = q_i, \quad i = 1, \dots, n \quad (17)$$

$$\tilde{q}_{2i} = f_i(q_1, \dots, q_{2n}, p_{n+1}, \dots, p_{2n}), \quad i = 1, \dots, n \quad (18)$$

$$\tilde{p}_{2i-1} = p_i - \frac{\partial F}{\partial q_i}(q_1, f_1(\dots), \dots, q_n, f_n(\dots), q_{n+1}, \dots, q_{2n}), \quad i = 1, \dots, n \quad (19)$$

$$\tilde{p}_{2i} = -\frac{\partial F}{\partial f_i}(q_1, f_1(\dots), \dots, q_n, f_n(\dots), q_{n+1}, \dots, q_{2n}), \quad i = 1, \dots, n \quad (20)$$

It is easily seen that the above transformation is a canonical one, i.e. the Poisson brackets are invariant. It is not hard to find the relevant generating function

$$\begin{aligned} \Phi(q_1, \dots, q_{2n}, \tilde{p}_1, \tilde{q}_2, \tilde{p}_3, \tilde{q}_4, \dots, \tilde{p}_{2n-1}, \tilde{q}_{2n}) &= \\ &= \sum_{k=1}^n q_k \tilde{p}_{2k-1} + F(q_1, \tilde{q}_2, q_2, \tilde{q}_4, \dots, q_n, \tilde{q}_{2n}, q_{n+1}, \dots, q_{2n}) \end{aligned} \quad (21)$$

It is also straightforward to check that both Hamiltonians coincide. Moreover, the definitions (17) ÷ (20) reduce on-shell to the Ostrogradski ones.

Let us now consider the case of Lagrangian depending on time derivatives up to some odd order

$$L = L(q, \dot{q}, \ddot{q}, \dots, q^{(2n+1)}) \quad (22)$$

Again, we define

$$q_i \equiv q^{(2i-2)}, \quad i = 1, \dots, n+1 \quad (23)$$

$$\dot{q}_i \equiv q^{(2i-1)}, \quad i = 1, \dots, n+1 \quad (24)$$

so that

$$L = L(q_1, \dot{q}_1, q_2, \dot{q}_2, \dots, q_{n+1}, \dot{q}_{n+1}) \quad (25)$$

Now, let us select a function F ,

$$F = F(q_1, \dot{q}_1, q_2, \dot{q}_2, \dots, q_n, \dot{q}_n, q_{n+1}, \dots, q_{2n+1}) \quad (26)$$

subject to the single condition

$$\det \left[\frac{\partial^2 F}{\partial q_i \partial \dot{q}_k} \right]_{\substack{n+2 \leq i \leq 2n+1 \\ 1 \leq k \leq n}} \neq 0 \quad (27)$$

and define the Lagrangian

$$\mathcal{L} = L + \sum_{k=1}^n \left(\frac{\partial F}{\partial q_k} \dot{q}_k + \frac{\partial F}{\partial \dot{q}_k} q_{k+1} \right) + \sum_{j=n+1}^{2n+1} \frac{\partial F}{\partial q_j} \dot{q}_j \quad (28)$$

Consider the Lagrange equations (8). First, we have

$$\sum_{k=1}^n \frac{\partial^2 F}{\partial q_i \partial \dot{q}_k} (q_{k+1} - \ddot{q}_k) = 0, \quad i = n+2, \dots, 2n+1 \quad (29)$$

and, by virtue of (27)

$$q_{k+1} = \ddot{q}_k, \quad k = 1, \dots, n \quad (30)$$

The remaining equations read

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) + \frac{\partial F}{\partial \dot{q}_{i-1}} - \frac{d^2}{dt^2} \left(\frac{\partial F}{\partial \dot{q}_i} \right) = 0, \quad i = 1, \dots, n+1 \quad (31)$$

with $\frac{\partial F}{\partial \dot{q}_0} = 0$, $\frac{\partial F}{\partial \dot{q}_{n+1}} = 0$. Combining (30) and (31) one gets

$$\sum_{k=0}^{2n+1} (-1)^k \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial q^{(k)}} \right) = 0 \quad (32)$$

Let us note that no condition of the form (5) is here necessary.

Also in the odd case the present formalism is related to that of Ostrogradski by a canonical transformation. Indeed, the canonical momenta read

$$\begin{aligned} p_i &= \frac{\partial F}{\partial q_i}, \quad i = n+2, \dots, 2n+1 \\ p_i &= \frac{\partial L}{\partial \dot{q}_i} + \sum_{k=1}^n \left(\frac{\partial^2 F}{\partial \dot{q}_i \partial q_k} \dot{q}_k + \frac{\partial^2 F}{\partial \dot{q}_i \partial \dot{q}_k} q_{k+1} \right) + \\ &+ \sum_{j=n+1}^{2n+1} \frac{\partial^2 F}{\partial \dot{q}_i \partial q_j} \dot{q}_j, \quad i = 1, \dots, n+1 \end{aligned} \quad (33)$$

and, as previously, these equations can be solved in terms of velocities, in particular

$$\dot{q}_i = f_i(q_1, \dots, q_{2n+1}, p_{n+2}, \dots, p_{2n+1}), \quad i = 1, \dots, n \quad (34)$$

Now, one can define the canonical transformation to Ostrogradski variables

$$\tilde{q}_{2i-1} = q_i, \quad i = 1, \dots, n+1 \quad (35)$$

$$\tilde{q}_{2i} = f_i(q_1, \dots, q_{2n+1}, p_{n+2}, \dots, p_{2n+1}), \quad i = 1, \dots, n \quad (36)$$

$$\tilde{p}_{2i-1} = p_i - \frac{\partial F}{\partial q_i}(q_1, f_1(\dots), \dots, q_n, f_n(\dots), q_{n+1}, \dots, q_{2n+1}), \quad i = 1, \dots, n+1 \quad (37)$$

$$\tilde{p}_{2i} = -\frac{\partial F}{\partial f_i}(q_1, f_1(\dots), \dots, q_n, f_n(\dots), q_{n+1}, \dots, q_{2n+1}), \quad i = 1, \dots, n \quad (38)$$

The relevant generating function reads

$$\begin{aligned} \Phi(q_1, q_2, \dots, q_{2n+1}, \tilde{p}_1, \tilde{q}_2, \tilde{p}_3, \tilde{q}_4, \dots, \tilde{q}_{2n}, \tilde{p}_{2n+1}) &= \\ &= \sum_{k=1}^{n+1} \tilde{p}_{2k-1} q_k + F(q_1, \tilde{q}_2, \dots, q_n, \tilde{q}_{2n}, q_{n+1}, \dots, q_{2n+1}) \end{aligned} \quad (39)$$

Sumarizing, we have found a modified Lagrangian and Hamiltonian formulations of higher-derivative theories. They are equivalent to the Ostrogradski formalism in the sense that on the Hamiltonian level they are related by a canonical transformation. However, the advantage of the approach presented is that the Lagrangian is nonsingular and the Legendre transformation can be performed in a straightforward way.

Let us conclude with the simple example. Consider the Lagrangian [6], [8], [9]

$$L = \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2}q^2 - gq\ddot{q} \quad (40)$$

and define

$$\begin{aligned} q_1 &= q, \quad q_2 = \ddot{q} \\ \mathcal{L} &= \frac{1}{2}\dot{q}_1^2 - \frac{\omega^2}{2}q_1^2 - gq_1q_2^2 + \frac{\partial F}{\partial q_1}\dot{q}_1 + \frac{\partial F}{\partial \dot{q}_1}q_2 + \frac{\partial F}{\partial q_2}\dot{q}_2 \end{aligned} \quad (41)$$

with F obeying

$$\frac{\partial F}{\partial \dot{q}_1} - 2gq_1q_2 = 0 \quad (42)$$

Eq.(42) implies

$$F(q_1, \dot{q}_1, q_2) = 2gq_1\dot{q}_1q_2 + W(q_1, q_2) \quad (43)$$

By neglecting total time derivative one obtains by virtue of eqs.(41), (43)

$$\mathcal{L} = \frac{1}{2}\dot{q}_1^2 - \frac{\omega^2}{2}q_1^2 + gq_1q_2^2 + 2g\dot{q}_1^2q_2 + 2gq_1\dot{q}_1\dot{q}_2 \quad (44)$$

It is now straightforward to construct the relevant Hamiltonian

$$H = \frac{p_1 p_2}{2gq_1} - \frac{(1 + 4gq_2)}{8g^2 q_1^2} p_2^2 + \frac{\omega^2}{2} q_1^2 - gq_1 q_2^2 \quad (45)$$

and one easily checks that the corresponding canonical equations yield the initial equation for $q \equiv q_1$.

The generating function for canonical transformation to Ostrogradski variables reads

$$\Phi(q_1, q_2, \tilde{p}_1, \tilde{q}_2) = q_1 \tilde{p}_1 + 2gq_1 q_2 \tilde{q}_2 \quad (46)$$

and gives

$$\begin{aligned} q_1 &= \tilde{q}_1 \\ q_2 &= -\frac{\tilde{p}_2}{2g\tilde{q}_1} \\ p_1 &= \tilde{p}_1 - \frac{\tilde{p}_2 \tilde{q}_2}{\tilde{q}_1} \\ p_2 &= 2g\tilde{q}_1 \tilde{q}_2 \end{aligned} \quad (47)$$

In terms of new variables the Hamiltonian (45) reads

$$H = \tilde{p}_1 \tilde{q}_2 - \frac{\tilde{p}_2^2}{4g\tilde{q}_1} - \frac{1}{2} \tilde{q}_2^2 + \frac{\omega^2}{2} \tilde{q}_1^2 \quad (48)$$

and coincides with the Ostrogradski Hamiltonian.

Let us note that both Hamiltonians (45) and (48) are singular for $g = 0$ and $q_1 (= \tilde{q}_1) = 0$. This is due to the fact that in this case the Lagrangian (40) reduces from second to first order in time derivatives.

Acknowledgement Thanks are due to Prof. Piotr Kosiński and Prof. Michal Majewski for interesting discussions and useful remarks.

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